THE ANTENNA LABORATORY

N64 13191

CODE-1

NASA CR 55141

22p.

RESEARCH ACTIVITIES in

OTS PRICE

XEROX $ 2.60 ph.

MICROFILM $ 0.86 mf.

UNPUBLISHED PRELIMINARY DATA

R. T. Compton, Jr.

Grant Number NsG-448

Prepared for

Department of ELECTRICAL ENGINEERING

THE OHIO STATE UNIVERSITY

RESEARCH FOUNDATION

Columbus, Ohio

(NASA CR-55141; Report 1691-2)

OTS: $2.60ph, $0.86mf

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REPORT
by
THE OHIO STATE UNIVERSITY RESEARCH FOUNDATION,
COLUMBUS, 12, OHIO

(2) Antenna Lab.

Sponsor: National Aeronautics and Space Administration, Washington 25, D. C.

Grant Number: (NASA Grant NsG-448)

Investigation of: Spacecraft Antenna Problems

Subject of Report: The Admittance of an Infinite Slot Radiating into a Lossy Half-Space

Submitted by: R. T. Compton, Jr., Antenna Laboratory, Department of Electrical Engineering

Date: 15 October 1963 22p refs

CONTENTS

THE ADMITTANCE OF AN INFINITE SLOT RADIATING INTO A LOSSY HALF-SPACE

I. INTRODUCTION

One method of measuring the properties of a plasma around a re-entry vehicle is to measure the admittance of an antenna which radiates in the presence of the plasma. To interpret such measurements, information is needed relating the admittance of the antenna to the plasma properties.

The problem of a dipole radiating into a lossy medium has been discussed by a number of authors.* Also, the properties of the loop antenna and the biconical antenna have been studied.* To the author's knowledge however, no work has been done to date on the admittances of aperture antennas radiating into lossy media. Since the aperture antenna is one of the most practical types for actual experiments, it is felt that a need exists for information on its properties.

In this report, the admittance of an infinite slot radiating into a lossy half-space is derived. The slot is assumed to be an opening in an infinitely conducting ground sheet. Outside the slot there is a semi-infinite region which is homogeneous and isotropic and has free-space permeability and arbitrary permittivity and conductivity.

The electric field in the slot is assumed to have the form of the TEM parallel-plate transmission line mode. The admittance per unit length of the slot is then calculated from the complex power flowing through the slot. The admittance is found as a function of the slot dimension, the frequency, and the constitutive parameters of the lossy region.

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* A bibliography of past work is given in:

"Antennas in Lossy Media," by C. T. Tai, to be published in the Proceedings of the 1963 URSI General Assembly. See also, "Radiation and Reception with Buried and Submerged Antennas," R.C.Hansen, IEEE Trans., Vol. AP-11, May 1963.

Although one obviously cannot put an infinite slot antenna on a satellite to make measurements, the results are still useful for the insight they provide. The infinite slot is probably the simplest antenna to handle analytically because the problem is two-dimensional; and in fact, as it turns out, the admittance properties of the infinite slot exhibit all the characteristics of other more practical geometries.

The problem treated here is intended to serve as a first approximation to the case of an electrically thick plasma slab.

II. DERIVATION OF RESULTS

Consider an infinite slot which opens through a ground sheet into a lossy half-space; as shown in Fig. 1. The slot extends infinitely far in the y-direction and has width "a" in the x-direction. Outside the slot, the xy-plane is infinitely conducting.

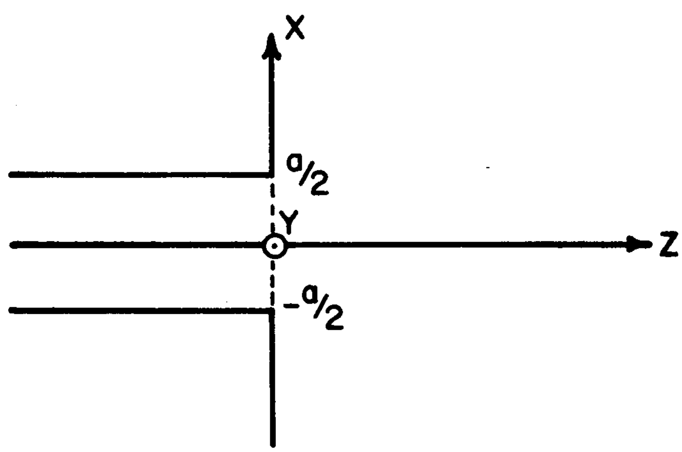

Fig. 1. Infinite slot in ground plane.

The half-space $z > 0$ is assumed to be homogeneous and isotropic, and is characterized by a complex propagation constant

$$(1) \qquad k = \left[\omega^2 \mu_o \epsilon \left(1 - i\frac{\sigma}{\omega \epsilon}\right)\right]^{\frac{1}{2}}$$

where

k = complex propagation constant

ω = radian frequency

μ_o = permeability of free-space

ϵ = permittivity of z > 0 region

σ = conductivity of z > 0 region.

MKS units and the time convention $e^{+i\omega t}$ will be used in this report.

The electric field in the slot is assumed to have the form of the TEM parallel-plate transmission line mode:

(2) $$E_x(x,y,o) = \begin{cases} 1 : |x| \leq \frac{a}{2} \\ 0 : |x| > \frac{a}{2} \end{cases} .$$

It is easily seen that the electromagnetic fields in this problem are everywhere TE to the y-axis. Hence the fields may be derived from a vector potential of the form

(3) $$\overline{F} = \hat{y}\,\psi$$

where ψ satisfies

(4) $$\nabla^2 \psi + k^2 \psi = 0$$

with appropriate boundary conditions. The electric and magnetic fields are related to $\overline{F}$ through

(5) $$\overline{E} = - \nabla \times \overline{F}$$

(6) $$\overline{H} = - (i\omega\epsilon + \sigma)\overline{F} + \frac{1}{i\omega\mu} \nabla (\nabla \cdot \overline{F}) .$$

Since the fields also have no y-dependence, the E_x and H_y components are given by

(7) $$E_x = + \frac{\partial \psi}{\partial z}$$

(8) $$H_y = - (i\omega\epsilon + \sigma)\,\psi .$$

A solution for ψ may be constructed of the form

(9) $$\psi(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k_x)\, e^{-ik_z z}\, e^{-ik_x x}\, dk_x$$

where

(10) $$k_z = \sqrt{k^2 - k_x^2}$$

and the root is chosen so that

(11) $$\mathrm{Re}(k_z) \geq 0$$

(12) $$\mathrm{Im}(k_z) \leq 0$$

corresponding to propagation in the +z-direction.

E_x is then given by

(13) $$E_x(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -ik_z f(k_x)\, e^{-ik_z z}\, e^{-ik_x x}\, dk_x,$$

or in the z = 0 plane

(14) $$E_x(x,o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} -ik_z f(k_x)\, e^{-ik_x x}\, dk_x \quad .$$

The function $f(k_x)$ is then found by taking the inverse transform:

(15) $$-ik_z\, f(k_x) = \int_{-\infty}^{\infty} E_x(x,o)\, e^{+ik_x x}\, dx \ .$$

Using (2), this gives

(16) $$-ik_z\, f(k_x) = \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{+ik_x x}\, dx = \frac{2}{k_x} \sin\left(\frac{k_x a}{2}\right)$$

or

(17) $$f(k_x) = \frac{2i}{k_x\, k_z} \quad \sin\left(\frac{k_x a}{2}\right) \qquad .$$

Thus the solution for ψ is:

(18) $$\psi(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2i}{k_x k_z} \sin\left(\frac{k_x a}{2}\right) e^{-ik_z z} e^{-ik_x x} dk_x \,,$$

and finally E_x and H_y are given by (7) and (8) as

(19) $$E_x(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{k_x} \sin\left(\frac{k_x a}{2}\right) e^{-ik_z z} e^{-ik_x x} dk_x \,,$$

(20) $$H_y(x,z) = \frac{-(i\omega\epsilon + \sigma)}{2\pi} \int_{-\infty}^{\infty} \frac{2i}{k_x k_z} \sin\left(\frac{k_x a}{2}\right) e^{-ik_z z} e^{-ik_x x} dk_x \,.$$

The complex power flowing through the aperture per unit length (per meter) in the y-direction is given by

(21) $$P = \frac{1}{2} \int_{-\frac{a}{2}}^{\frac{a}{2}} E_x(x,o)\, H_y^*(x,o)\, dx$$

where the asterick denotes the complex conjugate.

The admittance per unit length is then given by

(22) $$Y = \frac{2P^*}{|V|^2}$$

where V is the aperture voltage.

From (19) and (20),

(23) $$E_x(x,o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{k_x} \sin\left(\frac{k_x a}{2}\right) e^{-ik_x x} dk_x$$

(24) $$H_y(x,o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(\omega\epsilon - i\sigma)}{k_x\, k_z} \sin\left(\frac{k_x a}{2}\right) e^{-ik_x x} dk_x \quad .$$

Since $E_x(x,o)$ is zero for $|x| > \frac{a}{2}$, the limits of integration in (21) can be extended to infinity. Then substituting (23) and (24) in (21) and making use of Parseval's Theorem* gives for P^*,

(25) $$P^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(\omega\epsilon - i\sigma)}{k_x^2 \, k_z} \sin^2\left(\frac{k_x a}{2}\right) dk_x \quad .$$

Now the integrand may be rearranged and Parseval's Theorem may be used again. Let

(26) $$f_1(k_x) = \frac{2(\omega\epsilon - i\sigma)}{k_z}$$

and

(27) $$f_2(k_x) = \frac{1}{k_x^2} \sin^2\left(\frac{k_x a}{2}\right)$$

The transform of $f_1(k_x)$ is given by:

(28) $$F_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_x)\, e^{-ik_x x}\, dk_x$$

(29) $$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(\omega\epsilon - i\sigma)}{k_z} e^{-ik_x x}\, dk_x$$

* For the transform pairs

$$F_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_x)\, e^{-ik_x x}\, dk_x$$

$$F_2(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(k_x)\, e^{-ik_x x}\, dk_x$$

Parseval's Theorem is:

$$\int_{-\infty}^{\infty} F_1(x)\, F_2^*(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_x)\, f_2^*(k_x)\, dk_x \quad .$$

(30)* $$= (\omega \epsilon - i \sigma) H_0^{(2)} (k|x|)$$

The transform of $f_2(k_x)$ is

(31) $$F_2(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_2(k_x)\, e^{-ik_x x}\, dk_x$$

(32) $$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k_x^2} \sin^2 \left(\frac{k_x a}{2} \right) e^{-ik_x x}\, dk_x$$

(33) $$F_2(x) = \begin{cases} \frac{1}{4}(a - |x|) & : |x| \leq a \\ 0 & : |x| > a \end{cases} .$$

Then Parseval's Theorem gives

(34) $$P^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(k_x)\, f_2^*(k_x)\, dk_x = \int_{-\infty}^{\infty} F_1(x)\, F_2^*(x)\, dx$$

$$= \frac{(\omega \epsilon - i\sigma)}{4} \int_{-a}^{a} (a-|x|)\, H_0^{(2)}(k|x|)\, dx .$$

Since the integrand is an even function of x, this may be written

(35) $$P^* = \frac{\omega\epsilon - i\sigma}{2} \int_0^a (a-x)\, H_0^{(2)}(kx)\, dx .$$

* The integral in (29) is derived in:

R.T. Compton, Jr., "The Aperture Admittance of a Rectangular Waveguide Radiating into a Lossy Half-Space," Report 1691-1, Antenna Laboratory (In preparation).

The aperture voltage V is numerically equal to a. Hence from (22), the aperture admittance is given by:

$$(36) \qquad Y = \frac{\omega \epsilon - i\sigma}{a^2} \int_0^a (a-x)\, H_0^{(2)}(kx)\, dx \quad .$$

It is convenient at this point to normalize (36) with respect to the free-space constants. Let

$$(37) \qquad k_o = \omega \sqrt{\mu_o \epsilon_o} = \frac{2\pi}{\lambda_o}$$

be the free-space propagation constant, where ϵ_o is the permittivity of free-space and λ_o is the free-space wavelength for the frequency ω. Also let

$$(38) \qquad y_o = \sqrt{\frac{\epsilon_o}{\mu_o}}$$

be the characteristic admittance of free-space and define

$$(39) \qquad \eta = k_o x$$

$$(40) \qquad A = k_o a \quad .$$

Then (36) may be written in the form

$$(41) \qquad Y = \frac{k_o y_o}{A^2} \left(\frac{k}{k_o}\right)^2 \int_0^A (A-\eta)\, H_0^{(2)} \left[\left(\frac{k}{k_o}\right) \eta\right] d\eta$$

$$= \frac{k_o y_o}{A} \left(\frac{k}{k_o}\right)^2 \int_0^A H_0^{(2)} \left[\left(\frac{k}{k_o}\right) \eta\right] d\eta$$

$$- \frac{k_o y_o}{A^2} \left(\frac{k}{k_o}\right) \left[\eta\, H_1^{(2)} \left(\frac{k}{k_o}\, \eta\right)\right]_0^A$$

where $\eta\, H_1^{(2)}\left(\frac{k}{k_o}\,\eta\right)\Big|_{\eta=0}$ is understood to mean

(42) $$\eta\, H_1^{(2)}\left(\frac{k}{k_o}\,\eta\right)\Big|_{\eta=0} = \lim_{\eta\to o^+}\left[\eta\, H_1^{(2)}\left(\frac{k}{k_o}\,\eta\right)\right] .$$

Since $H_1^{(2)}(\rho) = J_1(\rho) - i\,N_1(\rho)$ and $J_1(o) = 0$,

(43) $$\lim_{\eta\to o^+}\left[\eta\, H_1^{(2)}\left(\frac{k}{k_o}\,\eta\right)\right] = -\,i\lim_{\eta\to o^+}\left[\eta\, N_1\left(\frac{k}{k_o}\,\eta\right)\right] .$$

For small ρ,

(44) $$N_1(\rho) = -\frac{2}{\pi}\,\frac{1}{\rho}\,J_o(\rho) + \frac{2}{\pi}\,\ln\frac{\gamma\rho}{2}\,J_1(\rho) - \frac{4}{\pi}\,\frac{\partial}{\partial\rho}\left[J_2(\rho) - \frac{1}{2}\,J_4(\rho) + \cdots\right]$$

where $\gamma = 1.781$, and thus

(45) $$\lim_{\eta\to o^+}\left[\eta\, H_1^{(2)}\left(\frac{k}{k_o}\,\eta\right)\right] = i\,\frac{2}{\pi}\,\frac{1}{k/k_o} .$$

Therefore (41) becomes:

(46) $$Y = \frac{k_o y_o}{A}\left(\frac{k}{k_o}\right)^2\int_o^A H_o^{(2)}\left[\left(\frac{k}{k_o}\right)\eta\right]d\eta - \frac{k_o y_o}{A}\left(\frac{k}{k_o}\right)H_1^{(2)}\left(\frac{k}{k_o}\,A\right) + i\,\frac{k_o y_o}{A^2}\,\frac{2}{\pi} .$$

Finally, with the substitution

(47) $$\frac{k}{k_o} = C\,e^{-i\phi}$$

(46) becomes

$$(48) \qquad Y = \frac{k_o y_o}{A^2} \, C\,A\, e^{-i2\phi} \int_o^{CA} H_o^{(2)} (\xi e^{-i\phi}) \, d\xi$$

$$- \frac{k_o y_o}{A^2} \, CA\, e^{-i\phi} \, H_1^{(2)} (CA\, e^{-i\phi}) + i \, \frac{k_o y_o}{A^2} \, \frac{2}{\pi} \quad .$$

As a check on the algebra, it may be seen that Y has the correct dimensions. A, C, and the integral are dimensionless. Hence Y has the dimensions of $k_o y_o$, i.e., mhos per meter.

Let

$$(49) \qquad I\,H_2\,(x, \alpha) = \int_o^x H_o^{(2)} (\xi\, e^{+i\alpha}) \, d\xi \quad .$$

Then Y, normalized to $(k_o y_o / A^2)$, is given by

$$(50) \qquad \frac{A^2}{k_o} \, \frac{Y}{y_o} = CA\, e^{-i\phi} \left[e^{-i\phi} \, IH_2 \, (CA, -\phi) - H_1^{(2)} (CA\, e^{-i\phi}) \right] + i \, \frac{2}{\pi} \quad .$$

Equation (50) has been evaluated* for $0 \leq CA \leq 10$ and $0^o \leq \phi \leq 90^o$, and is shown in Fig. 2. It is convenient to plot the normalized quantity $\frac{A^2}{k_o} \frac{Y}{y_o}$, since then only one curve need be drawn for all apertures.

---

* The function $I\,H_2(x, \alpha)$, and also the integrals

$$I\,J(x, \alpha) = \int_o^x J_o(\xi\, e^{+i\alpha})\, d\xi$$

$$I\,N(x, \alpha) = \int_o^x N_o(\xi\, e^{+i\alpha})\, d\xi$$

$$I\,H_1(x, \alpha) = \int_o^x H_o^{(1)}(\xi\, e^{+i\alpha})\, d\xi$$

have been tabulated by the author for $0 \leq x \leq 10.0$, $-90^o \leq \alpha \leq 90^o$, and will be published in a forthcoming Antenna Laboratory report.

The values of $H_1^{(2)}(z)$ for complex z were obtained from the following two tables:

(a) "Table of the Bessel Functions $J_o(z)$ and $J_1(z)$ for Complex Arguments," Mathematical Tables Project, National Bureau of Standards, Columbia University Press, New York, 1943.

(b) "Table of the Bessel Functions $Y_o(z)$ and $Y_1(z)$ for Complex Arguments," National Bureau of Standards, Columbia University Press, New York, 1950.

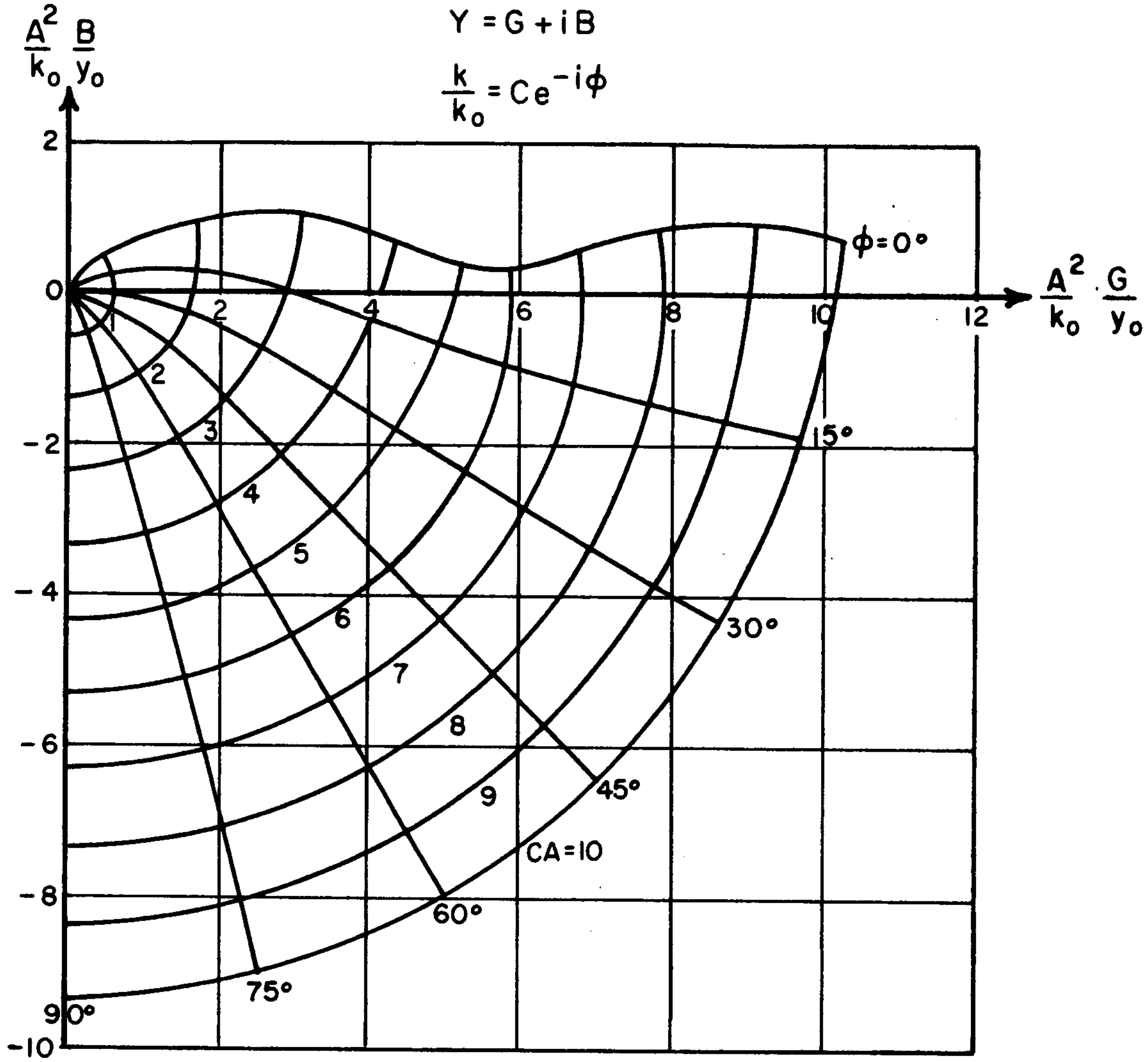

Fig. 2. The normalized admittance $\frac{A^2}{k_o} \frac{Y}{y_o}$.

For a fixed aperture size and fixed frequency, Fig. 2 shows the behavior of the admittance Y as a function of C, and hence as a function of ϵ and σ. *

If one is interested in the admittance of the slot as a function of the slot dimension A, the shape of the curves in Fig. 2 is misleading because of the factor A^2 in the normalizing constant for Y. This difficulty may be remedied by using the data of Fig. 2 to plot the quantity $Y/k_o y_o$ as a function of A (and ϕ) for fixed C.

For example, suppose we let $C = 1$. (Then $\phi = 0^o$ corresponds to free-space outside the aperture.) Then as a function of aperture size A, the normalized admittance has the behavior shown in Fig. 3.

Similarly, if one is interested in the dependence of Y on frequency for a given aperture and a given medium, the factor k_o must be removed from the normalization constant for Y in the same manner.

As a check on the numerical results shown in Fig. 2, it is helpful to consider two limiting cases. First, suppose the semi-infinite region has $\epsilon = \sigma = 0$ so that $C = 0$. (The point given by $CA = 0$ in Fig. 2 also corresponds to the d.c. case, i.e., $\omega = 0$, but the interpretation of the curves is somewhat tricky near $\omega = 0$. This case is discussed below.) Then it is clear from Eq. (24) that $H_y(x, 0) \equiv 0$ and therefore $Y = 0$. This accounts for the fact that the curves in Fig. 2 approach the origin as $C \to 0$.

Second, consider the case where C is large (and $\phi \neq 0$). Since the function $H_o^{(2)}\left(\frac{k}{k_o}\eta\right)$ decays rapidly to zero for complex $\frac{k}{k_o}$ as η becomes large, the integral in (48) may be replaced by

$$\int_o^{CA} H_o^{(2)}(\xi e^{-i\phi})\, d\xi \simeq \int_o^{\infty} H_o^{(2)}(\xi e^{-i\phi})\, d\xi \tag{51}$$

---

* It is interesting to note that the admittance of an infinite slot, as shown in Fig. 2, has the same general behavior as the admittance of a rectangular aperture radiating into a lossy medium. For curves of rectangular aperture admittance, see reference on page 7.

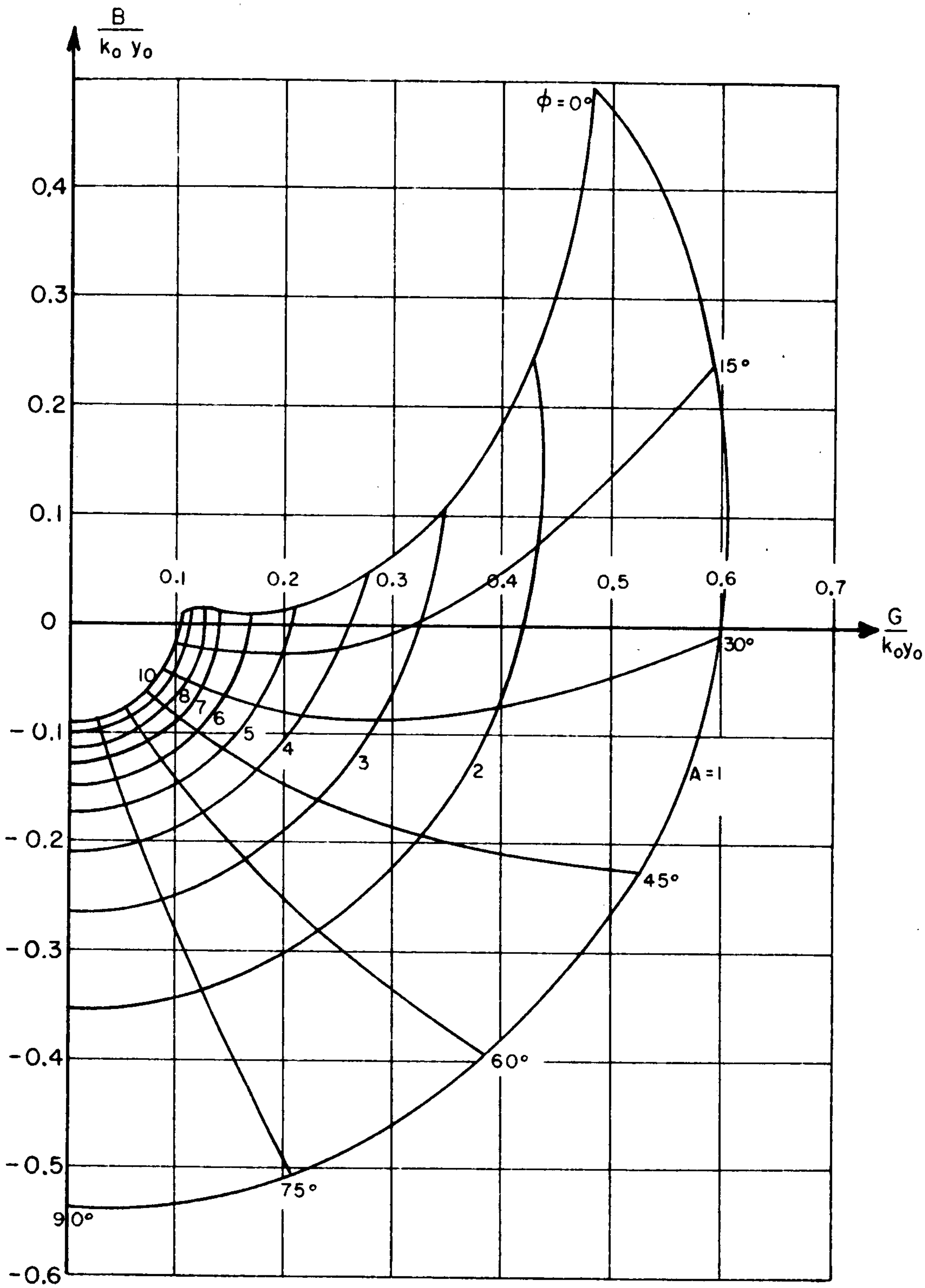

Fig. 3. Normalized admittance $\frac{Y}{k_o y_o}$ vs. "A" and "ϕ" for C = 1.

with little change in value. From the Fourier Transform pair*:

$$(52) \qquad \int_{-\infty}^{\infty} H_o^{(2)}(\alpha|x|)\, e^{+i k_x x}\, dx = \frac{2}{\sqrt{\alpha^2 - k_x^2}}$$

where

$$(53) \qquad \mathrm{Re}\left[\sqrt{\alpha^2 - k_x^2}\right] \geq 0$$

$$(54) \qquad \mathrm{Im}\left[\sqrt{\alpha^2 - k_x^2}\right] \leq 0$$

it is seen that

$$(55) \qquad \int_{o}^{\infty} H_o^{(2)}(\alpha x)\, dx = \frac{1}{\alpha} \qquad .$$

Hence for the integral in (51) we substitute

$$(56) \qquad \int_{o}^{CA} H_o^{(2)}(\xi e^{-i\phi})\, d\xi \simeq e^{i\phi} \qquad .$$

Also for large C,

$$(57) \qquad (CA)H_1^{(2)}(CA\, e^{-i\phi}) \simeq 0 \qquad (\phi \neq 0).$$

Making use of these gives the following approximate form for (48):

$$(58) \qquad Y \simeq \frac{k_o y_o}{A^2}\, CA\, e^{-i\phi} + i\, \frac{k_o y_o}{A^2}\, \frac{2}{\pi}$$

or

---

* See reference on page 7.

(59) $$\frac{A^2}{k_o}\ \frac{Y}{y_o} \simeq CA\ e^{-i\phi} + i\ \frac{2}{\pi} \quad .$$

This behavior is clearly indicated in Fig. 2. Comparison of Eq. (59) with the curves in Fig. 2 shows that (59) is quite accurate for $\phi > 15^{\circ}$ and $CA > 7$.

As mentioned above, the behavior of Y as a function of frequency is somewhat tricky as $\omega \to 0$. from (50), we have

(60) $$Y = \frac{k_o y_o\ C\ e^{-i\phi}}{A}\ \left[e^{-i\phi}\ I\ H_2\,(CA,\ -\phi) - H_1^{(2)}\ (CA\ e^{-i\phi})\right] + i\,\frac{2}{\pi}\ \frac{k_o y_o}{A^2} \quad .$$

For low frequencies, if $\sigma \neq 0$, C and CA are given by:

(61) $$C = \left|\frac{k}{k_o}\right| = \left|\frac{\sqrt{-i\omega\mu_o\,(i\omega\epsilon + \sigma)}}{\omega\sqrt{\mu_o\,\epsilon_o}}\right| \simeq \sqrt{\frac{\sigma}{\omega\,\epsilon_o}}$$

(62) $$CA = \left|\frac{k}{k_o}\right| k_o a = \sqrt{\omega\,\mu_o\,\sigma}\ \ a$$

Hence $CA \to 0$ as $\omega \to 0$. For small values of ρ,

(63) $$H_o^{(2)}(\rho) \simeq 1 - i\,\frac{2}{\pi}\ \ell n\ \frac{\gamma\rho}{2} \quad .$$

Therefore putting $\rho = \xi\, e^{i\alpha}$ and substituting (63) in (49) gives for small x

(64) $$IH_2\,(x, \alpha) \simeq x\left(1 + i\,\frac{2}{\pi} - i\,\frac{2}{\pi}\ \ell n\ \frac{\gamma x}{2} + \frac{2}{\pi}\ \alpha\right)$$

($\gamma = 1.781$). Similarly, for small ρ,

(65) $$H_1^{(2)}(\rho) \underset{-}{\sim} i \frac{2}{\pi} \frac{1}{\rho} - i \frac{\rho}{\pi} \ell n \frac{\gamma \rho}{2} .$$

Using (64) and (65) in (60) gives

(66) $$Y \underset{-}{\sim} \frac{k_o y_o C e^{-i\phi}}{A} \left[e^{-i\phi} CA \left(1 + i \frac{2}{\pi} - i \frac{2}{\pi} \ell n \frac{\gamma CA}{2} - \frac{2}{\pi} \phi \right) + i \frac{CA e^{-i\phi}}{\pi} \ell n \frac{\gamma CA}{2} + \frac{CA e^{-i\phi}}{\pi} \phi \right] .$$

Substituting (61) and (62) in and collecting terms gives for the leading term

(67) $$Y \underset{-}{\sim} -i \frac{\sigma}{\pi a} e^{-i2\phi} \ell n \frac{\gamma \sqrt{\omega \mu_o \sigma} a}{2} .$$

Since

(68) $$-\phi = \arg \sqrt{-i\omega \mu_o (i\omega \epsilon + \sigma)}$$

for small ω,

(69) $$\phi \underset{-}{\sim} \frac{\pi}{4}$$

and therefore

(70) $$Y \underset{-}{\sim} - \frac{\sigma}{\pi a} \ell n \frac{\gamma \sqrt{\omega \mu_o \sigma} a}{2} .$$

Because

(71) $$\lim_{\omega \to o^+} \ell n \frac{\gamma \sqrt{\omega \mu_o \sigma} a}{2} = - \infty$$

it is seen that $Y \to +\infty$ as $\omega \to 0$, for any $\sigma \neq 0$.

If $\sigma = 0$, however, instead of (61) and (62) we use

$$C = \left| \frac{k}{k_o} \right| = \sqrt{\frac{\epsilon}{\epsilon_o}} \tag{72}$$

$$CA = \omega \sqrt{\mu_o \epsilon} \; a \tag{73}$$

in (66). This gives

$$Y = \omega \epsilon \left[1 + i \frac{1}{\pi} \left(2 - \ell n \frac{\gamma \omega \sqrt{\mu_o \epsilon} \; a}{2} \right) \right] \tag{74}$$

which is an interesting result because

$$\lim_{\omega \to 0} Y = \begin{cases} +\infty : \sigma \neq 0 \\ 0 : \sigma = 0 \end{cases} . \tag{75}$$

This peculiar behavior may be understood by examining carefully the logarithmic term in (66). For any non-zero conductivity, this term contributes a singularity at $\omega = 0$. The lower the conductivity, the lower the frequency must be before this term contributes appreciably to Y. In the limit, for $\sigma = 0$, the singularity at $\omega = 0$ disappears.

Finally, it is interesting to make the following observation. Suppose the parallel-plate transmission line shown in Fig. 1, instead of feeding a semi-infinite half-space, feeds an infinite section of transmission line with the same dimensions and with a lossy dielectric between the plates, as shown in Fig. 4. The characteristic admittance per unit length in the y-direction of the line for $z > 0$ is

$$Y_c = \frac{1}{a} \sqrt{\frac{i\omega \epsilon + \sigma}{i \omega \mu_o}} . \tag{76}$$

The terminating admittance per unit length, Y', for the section of line $z < 0$ is simply Y_c. Hence, after some algebra

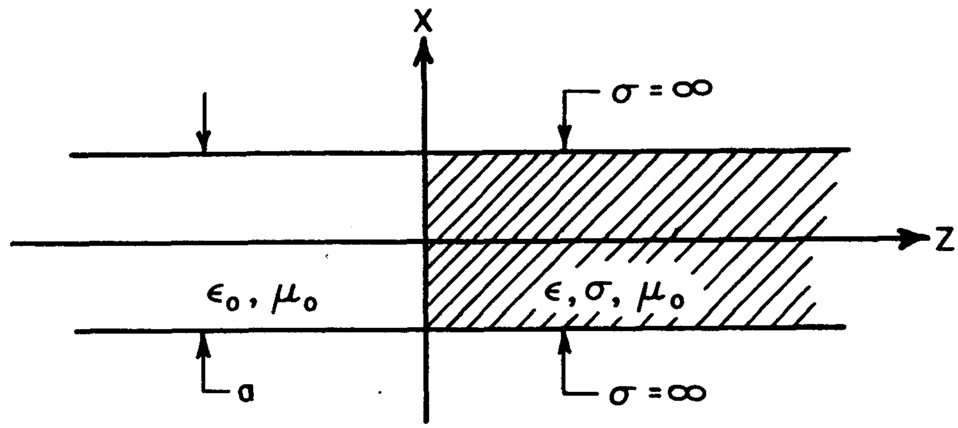

Fig. 4. Infinite transmission line model.

$$(77) \qquad \frac{A^2}{k_o}\,\frac{Y'}{y_o} = \frac{A^2}{k_o y_o}\,\frac{1}{a}\sqrt{\frac{i\,\omega\,\epsilon + \sigma}{i\,\omega\,\mu_o}} = CA\; e^{-i\phi}$$

which is the first term of (59). Thus, except for a constant $\left(i\,\frac{2}{\pi}\right)$, the admittance of the slot is correctly given by the model in Fig. 4, for large CA.

III. CONCLUSIONS

The admittance per unit length of an infinite slot radiating into an lossy half-space has been found and is shown in Fig. 2. Figure 2 is best interpreted as showing the admittance as a function of ϵ and σ, for fixed aperture size and fixed frequency. The dependence of Y on aperture size or on frequency can be calculated from Fig. 2. A sample curve of Y versus aperture size is shown in Fig. 3 for the case C = 1. Also it is noted that the behavior of Y near $\omega = 0$ is markedly different for $\sigma = 0$ than for $\sigma \neq 0$.

IV. ACKNOWLEDGMENT

The author is grateful to Professor Chen To Tai for many helpful discussions on this problem.